Landau energy levels for twisted N-enlarged Newton-Hooke space-time

Marcin Daszkiewicz

Institute of Theoretical Physics
University of Wroclaw pl. Maxa Borna 9, $50-206$ Wroclaw, Poland
$\mathrm{e-mail:}$ marcin@ift.uni.wroc.pl

Abstract

We derive the Landau energy levels for twisted N-enlarged Newton-Hooke space-time, i.e. we find the time-dependent energy spectrum and the corresponding eigenstates for an electron moving in uniform magnetic as well as in uniform electric external fields.

1 Introduction

The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on Quantum Gravity [2], [3] and String Theory models [4], [5], indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. Consequently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see e.g. [6]-[8]) as well as with field theoretical models (see e.g. [9]-[11]) in which the quantum space-time is employed.

It is well-known that a proper modification of the Poincare and Galilei Hopf algebras can be realized in the framework of Quantum Groups [12], [13]. Hence, in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [14], [15]), one can distinguish three types of quantum spaces [14], [15] (for details see also [16]):

1) Canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [17]-[19]

$$[\, x_\mu, x_\nu \,] = i\theta_{\mu\nu} \;, \tag{1}$$

2) Lie-algebraic modification of classical space-time [19]-[22]

$$[\, x_\mu, x_\nu \,] = i\theta_{\mu\nu}^{\rho} x_\rho \;, \tag{2}$$

and

3) Quadratic deformation of Minkowski and Galilei spaces [19], [22]-[24]

$$[\, x_\mu, x_\nu \,] = i\theta_{\mu\nu}^{\rho\tau} x_\rho x_\tau \;, \tag{3}$$

with coefficients $\theta_{\mu\nu}$, $\theta_{\mu\nu}^{\rho}$ and $\theta_{\mu\nu}^{\rho\tau}$ being constants.

Moreover, it has been demonstrated in [16], that in the case of so-called N-enlarged Newton-Hooke Hopf algebras $\mathcal{U}_0^{(N)}(NH_\pm)$ the twist deformation provides the new space-time noncommutativity of the form[1],[2]

$$\textbf{4)} \qquad [\, t, x_i \,] = 0 \quad , \quad [\, x_i, x_j \,] = i f_\pm\left(\frac{t}{\tau}\right)\theta_{ij}(x) \;, \tag{4}$$

with time-dependent functions

$$f_+\left(\frac{t}{\tau}\right) = f\left(\sinh\left(\frac{t}{\tau}\right), \cosh\left(\frac{t}{\tau}\right)\right) \quad , \quad f_-\left(\frac{t}{\tau}\right) = f\left(\sin\left(\frac{t}{\tau}\right), \cos\left(\frac{t}{\tau}\right)\right) \;,$$

[1] $x_0 = ct$.

[2] The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [17], [18]), for quantum N-enlarged Newton-Hooke Hopf algebras.

$\theta_{ij}(x) \sim \theta_{ij} = \text{const}$ or $\theta_{ij}(x) \sim \theta_{ij}^k x_k$ and τ denoting the time scale parameter - the cosmological constant. Besides, it should be noted that the mentioned above quantum spaces **1)**, **2)** and **3)** can be obtained by the proper contraction limit of the commutation relations **4)**[3].

As it was mentioned above, recently, there has been discussed the impact of different kinds of quantum spaces on the dynamical structure of physical systems (see e.g. [6]-[10]). However, the especially interesting results have been obtained in the series of papers [25]-[28] concerning the Hall effect for canonically deformed space-time (1). Particularly, there has been found the θ-dependent energy spectrum of an electron moving in uniform magnetic as well as in uniform electric field. Besides, it was demonstrated that in the case of noncommutative system one obtains the standard (commutative) Hall conductivity in terms of an effective magnetic field $B_{\text{eff}}(\theta)$.

In this article we derive the Landau energy levels for twisted N-enlarged Newton-Hooke space-time (11). Preciously, we find the time-dependent energy spectrum and the corresponding time-dependent eigenfunctions for an electron moving in uniform electric as well as magnetic field. Of course, for special choice of the noncommutativity function $f(t)$ (see formula (11)) i.e. for $f(t) = \theta$, we rediscovery partially the results of articles [25]-[28].

The paper is organized as follows. In Sect. 2 we recall basic facts concerning the twisted N-enlarged Newton-Hooke space-times provided in article [16]. The third section is devoted to the calculation of Landau energy levels for commutative (classical) space, while the Landau spectrum for twisted N-enlarged Newton-Hooke space-times is derived in Sect. 4. The final remarks are presented in the last section.

2 Twisted N-enlarged Newton-Hooke space-times

In this section we recall the basic facts associated with the twisted N-enlarged Newton-Hooke Hopf algebra $\mathcal{U}_\alpha^{(N)}(NH_\pm)$ and with the corresponding quantum space-times [16]. Firstly, it should be noted that in accordance with Drinfeld twist procedure the algebraic sector of twisted Hopf structure $\mathcal{U}_\alpha^{(N)}(NH_\pm)$ remains undeformed, i.e. it takes the form

$$[\, M_{ij}, M_{kl}\,] = i\,(\delta_{il}\, M_{jk} - \delta_{jl}\, M_{ik} + \delta_{jk} M_{il} - \delta_{ik} M_{jl}) \quad , \quad [\, H, M_{ij}\,] = 0 \;, \tag{5}$$

$$\left[\, M_{ij}, G_k^{(n)}\,\right] = i\left(\delta_{jk}\, G_i^{(n)} - \delta_{ik}\, G_j^{(n)}\right) \quad , \quad \left[\, G_i^{(n)}, G_j^{(m)}\,\right] = 0 \;, \tag{6}$$

$$\left[\, G_i^{(k)}, H\,\right] = -ikG_i^{(k-1)} \quad , \quad \left[\, H, G_i^{(0)}\,\right] = \pm\frac{i}{\tau}G_i^{(1)} \quad ; \quad k > 1 \;, \tag{7}$$

where τ, M_{ij}, H, $G_i^{(0)}(= P_i)$, $G_i^{(1)}(= K_i)$ and $G_i^{(n)}(n > 1)$ can be identified with cosmological time parameter, rotation, time translation, momentum, boost and accelerations

[3] Such a result indicates that the twisted N-enlarged Newton-Hooke Hopf algebra plays a role of the most general type of quantum group deformation at nonrelativistic level.

operators respectively. Besides, the coproducts and antipodes of considered algebra are given by[4]

$$\Delta_\alpha(a) = \mathcal{F}_\alpha \circ \Delta_0(a) \circ \mathcal{F}_\alpha^{-1} \quad , \quad S_\alpha(a) = u_\alpha \, S_0(a) \, u_\alpha^{-1} \, , \tag{8}$$

with $u_\alpha = \sum f_{(1)} S_0(f_{(2)})$ (we use Sweedler's notation $\mathcal{F}_\alpha = \sum f_{(1)} \otimes f_{(2)}$) and with the twist factor $\mathcal{F}_\alpha \in \mathcal{U}_\alpha^{(N)}(NH_\pm) \otimes \mathcal{U}_\alpha^{(N)}(NH_\pm)$ satisfying the classical cocycle condition

$$\mathcal{F}_{\alpha 12} \cdot (\Delta_0 \otimes 1) \; \mathcal{F}_\alpha = \mathcal{F}_{\alpha 23} \cdot (1 \otimes \Delta_0) \; \mathcal{F}_\alpha \, , \tag{9}$$

and the normalization condition

$$(\epsilon \otimes 1) \; \mathcal{F}_\alpha = (1 \otimes \epsilon) \; \mathcal{F}_\alpha = 1 \, , \tag{10}$$

such that $\mathcal{F}_{\alpha 12} = \mathcal{F}_\alpha \otimes 1$ and $\mathcal{F}_{\alpha 23} = 1 \otimes \mathcal{F}_\alpha$.

The corresponding quantum space-times are defined as the representation spaces (Hopf modules) for N-enlarged Newton-Hooke Hopf algebra $\mathcal{U}_\alpha^{(N)}(NH_\pm)$. Generally, they are equipped with two spatial directions commuting to classical time, i.e. they take the form

$$[\, t, \hat{x}_i \,] = [\, \hat{x}_1, \hat{x}_3 \,] = [\, \hat{x}_2, \hat{x}_3 \,] = 0 \quad , \quad [\, \hat{x}_1, \hat{x}_2 \,] = if(t) \;\; ; \;\; i = 1,2,3 \, . \tag{11}$$

However, it should be noted that this type of noncommutativity has been constructed explicitly only in the case of 6-enlarged Newton-Hooke Hopf algebra, with [16][5]

$$\begin{aligned}
f(t) &= f_{\kappa_1}(t) = f_{\pm,\kappa_1}\left(\frac{t}{\tau}\right) = \kappa_1 \, C_\pm^2\left(\frac{t}{\tau}\right) \, , \\
f(t) &= f_{\kappa_2}(t) = f_{\pm,\kappa_2}\left(\frac{t}{\tau}\right) = \kappa_2 \tau \, C_\pm\left(\frac{t}{\tau}\right) S_\pm\left(\frac{t}{\tau}\right) \, , \\
&\;\;\vdots \\
f(t) &= f_{\kappa_{35}}\left(\frac{t}{\tau}\right) = 86400 \kappa_{35} \, \tau^{11} \left(\pm C_\pm\left(\frac{t}{\tau}\right) \mp \frac{1}{24}\left(\frac{t}{\tau}\right)^4 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp 1 \right) \times \\
&\quad \times \left(S_\pm\left(\frac{t}{\tau}\right) \mp \frac{1}{6}\left(\frac{t}{\tau}\right)^3 - \frac{t}{\tau} \right) , \\
f(t) &= f_{\kappa_{36}}\left(\frac{t}{\tau}\right) = 518400 \kappa_{36} \, \tau^{12} \left(\pm C_\pm\left(\frac{t}{\tau}\right) \mp \frac{1}{24}\left(\frac{t}{\tau}\right)^4 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp 1 \right)^2 ,
\end{aligned} \tag{12}$$

[4] $\Delta_0(a) = a \otimes 1 + 1 \otimes a$, $S_0(a) = -a$.

[5] $\kappa_a = \alpha$ ($a = 1, ..., 36$) denote the deformation parameters.

and

$$C_{+/-}\left(\frac{t}{\tau}\right) = \cosh / \cos\left(\frac{t}{\tau}\right) \quad \text{and} \quad S_{+/-}\left(\frac{t}{\tau}\right) = \sinh / \sin\left(\frac{t}{\tau}\right) \; .$$

Besides, one can easily check that in τ approaching infinity limit the above quantum spaces reproduce the canonical (1), Lie-algebraic (2) and quadratic (3) type of space-time noncommutativity, i.e. for $\tau \to \infty$ we get

$$\begin{aligned} f_{\kappa_1}(t) &= \kappa_1 \; , \\ f_{\kappa_2}(t) &= \kappa_2 \, t \; , \\ &\cdot \\ &\cdot \\ &\cdot \\ f_{\kappa_{35}}(t) &= \kappa_{35} \, t^{11} \; , \\ f_{\kappa_{36}}(t) &= \kappa_{36} \, t^{12} \; . \end{aligned} \tag{13}$$

Of course, for all deformation parameters $\alpha = \kappa_a$ running to zero the above deformations disappear.

3 Landau energy levels for commutative space-time

In this section we turn to the derivation of Landau energy levels for commutative (classical) space. Firstly, it should be noted, that in such a case the algebra of position and momentum operators takes the form

$$[\, x_i, x_j \,] = 0 = [\, p_i, p_j \,] \quad , \quad [\, x_i, p_j \,] = i\hbar\delta_{ij} \; . \tag{14}$$

Further, we define the following Hamiltonian function

$$H = H(\bar{p}, \bar{x}) = \frac{1}{2m}\left(\bar{p} + \frac{e}{c}\bar{A}(\bar{x})\right)^2 - e\phi(\bar{x}) \; , \tag{15}$$

which describes an electron moving in the (x_1, x_2)-plane in the uniform external electric field $\bar{E} = -\mathrm{grad}\phi$ and in the uniform perpendicular to plane magnetic field $\bar{B} = \mathrm{rot}\bar{A}$. Besides, in our considerations we adopt the so-called symmetric gauge

$$\bar{A}(\bar{x}) = \left[-\frac{B}{2}x_2, \frac{B}{2}x_1\right] \; , \tag{16}$$

as well as we take

$$\phi(\bar{x}) = -Ex_1 \; . \tag{17}$$

Consequently, we get

$$H(\bar{p}, \bar{x}) = \frac{1}{2m}\left[\left(p_1 - \frac{eB}{2c}x_2\right)^2 + \left(p_2 + \frac{eB}{2c}x_1\right)^2\right] + eEx_1 \; . \tag{18}$$

Let us now turn to the eigenvalue problem for hamiltonian function (18) encoded by

$$H(\bar{p},\bar{x})\psi(\bar{x}) = \mathcal{E}\psi(\bar{x}) \; . \tag{19}$$

In order to solve the above equation we perform the following change of variables

$$x = x_1 + ix_2 \quad , \quad p = \frac{1}{2}(p_1 - ip_2) \; , \tag{20}$$

as well as we introduce two families of creation/annihilation operators

$$a^{\dagger} = -2ip^{*} + \frac{eB}{2c}x + \lambda \quad , \quad a = 2ip + \frac{eB}{2c}x^{*} + \lambda \; , \tag{21}$$

and

$$b = 2ip - \frac{eB}{2c}x^{*} \quad , \quad b^{\dagger} = -2ip^{*} - \frac{eB}{2c}x \quad ; \quad (\alpha + i\beta)^{*} = \alpha - i\beta \; , \tag{22}$$

with parameter $\lambda = \frac{mcE}{B}$. One can easily check that these two sets commute with each other and satisfy the following commutation relations

$$[\, a, a^{\dagger} \,] = 2m\hbar\omega \quad , \quad [\, b^{\dagger}, b \,] = 2m\hbar\omega \; , \tag{23}$$

with $\omega = \frac{eB}{mc}$ denoting cyclotron frequency. Besides, we observe that the Hamiltonian $H(\bar{p},\bar{x})$ can be written as

$$H(a,b) = \frac{1}{4m}\left(a^{\dagger}a + aa^{\dagger}\right) - \frac{\lambda}{2m}\left(b^{\dagger} + b\right) - \frac{\lambda^2}{2m} \; . \tag{24}$$

In order to find the eigenvalues $\mathcal{E}$ and eigenfunctions $\psi(\bar{x})$ we separate the operator (24) into two mutually commuting parts

$$H(a,b) = H(a) + H(b) \; , \tag{25}$$

where $H(a)$ denotes the harmonic oscillator part

$$H(a) = \frac{1}{4m}\left(a^{\dagger}a + aa^{\dagger}\right) \; , \tag{26}$$

while $H(b)$ - the part linear in b and $b^{\dagger}$ operators, given by

$$H(b) = \frac{\lambda}{2m}\left(b^{\dagger} + b\right) + \frac{\lambda^2}{2m} \; . \tag{27}$$

In the case of the harmonic oscillator part

$$H(a)\psi_n = \mathcal{E}_n\psi_n \; , \tag{28}$$

one can proceed in the standard way and gets the following discrete spectrum

$$\phi_n = \frac{1}{\sqrt{(2m\hbar\omega)^n n!}}(a^\dagger)^n|0> \quad , \quad \mathcal{E}_n = \frac{\hbar\omega}{2}(2n+1) \ ; \ n = 0, 1, 2\ldots \ , \tag{29}$$

with vacuum state $|0>$ such that $a|0>= 0$. For the eigenvalue equation

$$H(b)\psi = \mathcal{E}\psi \ , \tag{30}$$

the situation seems to be more complicated. However, by simple calculation we find the following continuous spectrum

$$\psi_\alpha = \exp i\left(\alpha x_2 + \frac{m\omega}{2\hbar}x_1x_2\right) \quad , \quad \mathcal{E}_\alpha = \frac{\hbar\lambda}{m}\alpha + \frac{\lambda^2}{2m} \ , \tag{31}$$

with real parameter α. Consequently, the eigenfunctions and the energy spectrum of the whole Hamiltonian $H(a,b)$ are given by

$$\psi_{(n,\alpha)}(\bar{x}) = \psi_n \otimes \psi_\alpha \quad , \quad \mathcal{E}_{(n,\alpha)} = \frac{\hbar\omega}{2}(2n+1) - \frac{\hbar\lambda}{m}\alpha - \frac{\lambda^2}{2m} \ , \tag{32}$$

where $n = 0, 1, 2\ldots$, $\alpha \in \mathbb{R}$ and where symbol $\otimes$ denotes the direct product of two wave functions. The formula (32) defines so-called Landau energy levels for commutative nonrelativistic space-time (14).

4 Landau energy levels for twisted N-enlarged Newton-Hooke space-time

Let us now turn to the main aim of our investigations - to the derivation of Landau energy levels for quantum space-times (11). In first step of our calculation we extend the described in second section spaces to the whole algebra of momentum and position operators as follows

$$[\ \hat{x}_1, \hat{x}_2\ ] = if_{\kappa_a}(t) \quad , \quad [\ \hat{p}_i, \hat{p}_j\ ] = 0 \quad , \quad [\ \hat{x}_i, \hat{p}_j\ ] = i\hbar\delta_{ij} \ . \tag{33}$$

One can check that relations (33) satisfy the Jacobi identity and for deformation parameters κ_a approaching zero become classical.
Next, by analogy to the commutative case (see formulas (15), (16)) we define the following Hamiltonian operator[6],[7]

$$\hat{H} = \hat{H}(\bar{\hat{p}}, \bar{\hat{x}}) = \frac{1}{2m}\left(\bar{\hat{p}} + \frac{e}{c}\bar{\hat{A}}(\bar{\hat{x}})\right)^2 - e\hat{\phi}(\bar{\hat{x}}) \ . \tag{34}$$

[6]The choice of the Hamiltonian (34) (similar to the classical operator (15)) permits to investigate the (direct) impact of noncommutative space-time on the dynamical structure of considered system. As we shall see for a moment such a defined Hamiltonian in terms of the commutative position operators becomes time-dependent (see formula (36)). This situation is interpreted by the author as the generating by space-time noncommutativity (11) of additional (time-dependent) interaction of particle with some external source.

[7]We define the model only in noncommutative $(\hat{x}_1, \hat{x}_2)$-plane, i.e. $\bar{\hat{p}} = (\hat{p}_1, \hat{p}_2)$ and $\bar{\hat{x}} = (\hat{x}_1, \hat{x}_2)$.

In order to analyze the above system we represent the noncommutative operators $(\hat{x}_i, \hat{p}_i)$ by classical ones (x_i, p_i) as (see e.g. [29])

$$\hat{x}_1 = x_1 - \frac{f_{\kappa_a}(t)}{2\hbar} p_2 \quad , \quad \hat{x}_2 = x_2 + \frac{f_{\kappa_a}(t)}{2\hbar} p_1 \quad , \quad \hat{p}_i = p_i \ . \tag{35}$$

Then, the Hamiltonian (34) in the symmetric gauge (16) takes the form

$$\begin{aligned} \hat{H}(t) &= \frac{1}{2m}\left[\left((1-\alpha_{\kappa_a}(t))p_1 - \frac{eB}{2c}x_2\right)^2 + \left((1-\alpha_{\kappa_a}(t))p_2 + \frac{eB}{2c}x_1\right)^2\right] + \\ &+ eE\left(x_1 - \frac{f_{\kappa_a}(t)}{2\hbar}p_2\right) = \hat{H}(\bar{p}, \bar{x}, t) \ , \end{aligned} \tag{36}$$

with function $\alpha_{\kappa_a}(t) = \frac{ef_{\kappa_a}(t)B}{4\hbar c}$.

In order to solve the eigenvalue problem

$$\hat{H}(t)\psi = \mathcal{E}\psi \ , \tag{37}$$

we introduce (similar to the commutative case) two sets of time-dependent operators

$$a^\dagger(t) = -2i\bar{p}^*(t) + \frac{eB}{2c}x + \lambda_-(t) \quad , \quad a(t) = 2i\bar{p}(t) + \frac{eB}{2c}x^* + \lambda_-(t) \ , \ , \tag{38}$$

as well as

$$b(t) = 2i\bar{p}(t) - \frac{eB}{2c}x^* \quad , \quad b^\dagger(t) = -2i\bar{p}^*(t) - \frac{eB}{2c}x \ , \tag{39}$$

where $\bar{p}(t) = \beta(t)p = (1-\alpha_{\kappa_a}(t))p$ and where $\lambda_-(t)$ denotes the real and arbitrary function which will be fixed for a moment. One can check that both families of operators $(a(t), a^\dagger(t))$ and $(b(t), b^\dagger(t))$ commute each other and satisfy the following commutation relations[8]

$$[\ a(t), a^\dagger(t)\ ] = 2m\hbar\omega(t) \quad , \quad [\ b^\dagger(t), b(t)\ ] = 2m\hbar\omega(t) \ , \tag{40}$$

with $\omega(t) = \beta(t)\omega$. Moreover, it is easy to see that Hamiltonian $\hat{H}(t)$ can be written as

$$\hat{H}(a(t), b(t)) = \frac{1}{4m}\left(a^\dagger(t)a(t) + a(t)a^\dagger(t)\right) - \frac{\lambda_+(t)}{2m}\left(b^\dagger(t) + b(t)\right) - \frac{\lambda_-^2(t)}{2m} \ , \tag{41}$$

where functions $\lambda_\pm(t)$ are fixed to be

$$\lambda_\pm(t) = \lambda \pm \frac{emEf_\kappa(t)}{4\beta(t)\hbar} \ . \tag{42}$$

[8]We perform our calculations for such times that $\omega(t) > 0$. Such a situation appears for "almost" all times for $0 < \kappa_a << 1$.

The solution of eigenvalue problem (37) can be found by direct calculation; it looks as follows[9]

$$
\begin{aligned}
\psi_{(n,\alpha,\kappa_a)}(t) &= \frac{1}{\sqrt{(2m\hbar\omega(t))^n n!}} \exp i\left(\alpha x_2 + \frac{m\omega(t)}{2\hbar\beta^2(t)} x_1 x_2\right)(a^\dagger(t))^n|0> \,, \\
\mathcal{E}_{(n,\alpha,\kappa_a)}(t) &= \frac{\hbar\omega(t)}{2}(2n+1) - \frac{\hbar\beta(t)\lambda_+(t)}{m}\alpha - \frac{m}{2}\lambda_-^2(t) \,, \qquad (43)
\end{aligned}
$$

with $n = 0, 1, 2\ldots$ and $\alpha \in \mathbb{R}$. The formula (43) defines the Landau energy levels for twisted N-enlarged Newton-Hooke space-time (11). Besides, it should be noted, that for $f_{\kappa_a}(t) = \theta$ we recover the energy spectrum for canonical deformation derived in paper [25], while for all parameters κ_a approaching zero the above results become the same as in commutative case, i.e. we get the formula (32).

5 Final remarks

In this article we derive the Landau energy levels for twisted N-enlarged Newton-Hooke space-time. Preciously, we find the time-dependent energy spectrum for an electron moving in uniform magnetic as well as in uniform electric (external) fields. It should be also mentioned that the presented investigation has been performed for the quite general (constructed explicitly) type of space-time noncommutativity at nonrelativistic level.

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